Eur. Phys. J. C 51, 241–247 (2007)
DOI 10.1140/epjc/s10052-007-0297-4

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Dirac equation studies in the tunneling energy zone

S. De Leo[1,a], P.P. Rotelli[2,b]

[1] Department of Applied Mathematics, State University of Campinas, PO Box 6065, SP 13083-970, Campinas, Brazil
[2] Department of Physics, INFN, University of Lecce, PO Box 193, 73100, Lecce, Italy

Received: 22 September 2006 / Revised version: 1 February 2007 /
Published online: 28 April 2007 – © Springer-Verlag / Società Italiana di Fisica 2007

Abstract. We investigate the tunneling zone $V_0 - m < E < V_0 + m$ for a one-dimensional potential in the Dirac equation. We find the appearance of superluminal transit times, akin to the Hartman effect.

PACS. 03.65.Pm; 03.65.Xp

1 Introduction

In several recent articles, we have investigated in some detail one-dimensional electrostatic potentials by means of both the Schrödinger [1–3] and the Dirac equation [4–6]. Several original phenomena have been observed, such as the transition from resonance phenomena to multiple (infinite) peak formation [5], equivalent to a shift from wave-like to particle behavior in the barrier diffusion zone $E > V_0 + m$, where V_0 is the barrier height, E one of the wave packet energies and m the particle mass. We have also investigated the compatibility of the barrier results with the Klein paradox [7–9], when $m < E < V_0 - m$. In this latter case, we have noted the existence of dynamic localized states with a continuous spectrum [6]. These states are the nearest approximation to the bound states of Schrödinger or of Dirac in the evanescent energy zone considered in this paper.

The evanescent zone is the last energy zone we need to consider to complete our analysis. It is given by $V_0 - m < E < V_0 + m$ ($V_0 > 2m$) or $m < E < V_0 + m$ ($V_0 < 2m$). It has evanescent (real exponential) space forms in the barrier region. For a well potential, it is just such forms that give rise to discrete bound states. In this paper, we shall concentrate on the single barrier potential and hence complete our analysis for this elementary structure. The evanescent stationary solutions become dynamic if instead of plane waves we work with incoming wave packets. Then the particles within the classically forbidden region are measurable only for a finite time, the time of transition from an incoming wave packet to reflected/transmitted wave packets. Even for the step potential in this energy zone there will exist, during this transitory time, a current flow both *into* and subsequently *out of* the step. Since the stationary solution has a zero current flow within the step, this feature is not always recognized. It is, however, obvious when one admits that there will be a nonzero transitory $\mathrm{d}\rho/\mathrm{d}t$ for any space interval within the step.

The most important barrier feature (both theoretically and experimentally) of this energy zone is tunneling. A part of the incoming wave packet will continue its course beyond the barrier region. Its magnitude will be modulated by the barrier. For "large" barriers (compared to the wave packet size) an exponential reduction in amplitude $\propto \exp[-ql]$ occurs, where l is the barrier length and q is $\sqrt{m^2 - (E - V_0)^2}$. This not only reduces the amplitude but modifies the transition spectrum. The smaller the transition amplitude is, the smaller the modifications in the *reflected wave packet* from the incident wave packet. However, in general, for both the reflected and transmitted wave packets we have maxima in configuration space and can apply the stationary phase method (SPM) to calculate the reflection time delays and transition times [10]. With the Schrödinger equation the conclusion that the transition time is *independent* of the barrier width l, when $l \to \infty$, is known as the Hartman effect [11]. Such a result is hard to avoid and, if the same occurs for the Dirac equation (the subject matter of this paper, previously discussed by other authors [12–14]), we would have to face the unpalatable feature of superluminal velocities within the barrier. We warn that more than one type of transition time has been defined in the literature [10, 15–17], and for details we refer to [18, 19]. In this paper, we intend to investigate this particular aspect of tunneling by means of the SPM, neglecting the possible ambiguities that this approximation is known to have.

In the next section, we define all quantities and equations used. Some of these have been given also elsewhere [5, 6], but for completeness we present them again. We also solve the stationary plane wave problem for the step and barrier. In Sect. 3, we calculate the transition

[a] e-mail: deleo@ime.unicamp.br
[b] e-mail: rotelli@le.infn.it

times by using the SPM and, based on numerical calculations, we discuss the appearance of superluminal velocities. Our conclusions are drawn in Sect. 4.

2 Formalism and solutions for the barrier

We shall work with a one-dimensional (electrostatic) potential in the Dirac equation. The chosen axis is the z-axis. However, we shall use the solutions and hence the formalism of the full three-dimensional case [21]. Thus, the stationary Dirac equation reads

$$[E - V(z)]\gamma_0\psi(z) + \mathrm{i}\gamma_3\psi'(z) = m\psi(z)\,, \tag{1}$$

where γ_0 and γ_3 are two of the Dirac matrices (see below) and $\psi'(z) = \mathrm{d}\psi(z)/\,\mathrm{d}z$. The explicit time dependence $\exp[-\mathrm{i}Et]$ has been dropped from the above equation, and hence ψ is only a function of the z coordinate. Our chosen representation for the gamma matrices is the Pauli–Dirac one, so that

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}. \tag{2}$$

The barrier potential is fixed at V_0 in the region $0 < z < l$ and is zero elsewhere. The z-axis is divided into three regions. Region I is the region of the incident and reflected waves ($z < 0$). Region II is the barrier region. Region III contains the transmitted wave ($z > l$). For $V_0 > m$, it is convenient to divide the tunnel energy zone into two subzones, both evanescent: (A) $V_0 < E < V_0 + m$, and (B) $V_0 - m < E < V_0$ ($V_0 > 2m$) or $m < E < V_0$ ($m < V_0 < 2m$). For $V_0 < m$, only the evanescent zone (A), $m < E < V_0 + m$, exists. The A zone corresponds to an energy larger than the potential ($E > V_0$), and we will use the "positive energy" $u^{(s)}$ solutions modified for a non-zero potential V_0. The B zone corresponds to the zone with an energy below the potential ($E < V_0$).

One of the questions we ask in this work is if all solutions with energy less than the potential ("negative energy") represent physical antiparticles, be they oscillatory (free) or evanescent. For the B zone we shall use the $u^{(s+2)}$ solutions modified to allow for a constant non-zero potential V_0.

Spin flip is absent in all these problems (independent of the value of E), so by choosing an incoming spin up state,

$$u^{(1)}(p,E) = [1, 0, p/(E+m), 0]^{\mathrm{t}}\,, \tag{3}$$

we find the following spinors in region II:

$$\begin{aligned} &\text{A zone}: \\ &u^{(1)}(\pm\mathrm{i}q, E - V_0) = [1, 0, \pm\mathrm{i}q/(E - V_0 + m), 0]^{\mathrm{t}}\,, \\ &\text{B zone}: \\ &u^{(3)}(\pm\mathrm{i}q, |E - V_0|) = [\mp\mathrm{i}q/(|E - V_0| + m), 0, 1, 0]^{\mathrm{t}}\,. \end{aligned} \tag{4}$$

Only the spinor $u^{(1)}(p,E)$ appears in region III.

2.1 A zone: $V_0 < E < V_0 + m$ ($V_0 > m$) or $m < E < V_0 + m$ ($V_0 < m$)

The solutions in the three regions are

$$\begin{aligned} &\text{Region I}: z < 0\,, \\ &u^{(1)}(p,E)\exp[\mathrm{i}pz] + R_{>}u^{(1)}(-p,E)\exp[-\mathrm{i}pz]\,, \\ &\text{Region II}: 0 < z < l\,, \\ &A_{>}u^{(1)}(\mathrm{i}q, E - V_0)\exp[-qz] + B_{>}u^{(1)}(-\mathrm{i}q, E - V_0) \\ &\quad \times \exp[qz]\,, \\ &\text{Region III}: l < z\,, \\ &T_{>}u^{(1)}(p,E)\exp[\mathrm{i}pz]\,, \end{aligned} \tag{5}$$

and we are using un-normalized solutions, but such that $|R_{>}|^2$ is the reflection probability. The symbols $R_{\lessgtr}$ ($T_{\lessgtr}$) will be used for the A/B energy zones, because $E \lessgtr V_0$, respectively. The solutions in region II are the evanescent solutions $\propto \exp[\pm qz]$. We shall in the following refer to the case of a step potential with only two regions (I and II), without treating this case separately; we merely note that it corresponds to the above solutions with $B_{\lessgtr} = 0$. It should also be obtainable from the barrier solution when $l \to \infty$, although some care must be taken when multiple peaks occur, such as in the case of diffusion above potential [5]. The first of these barrier peaks reproduces the single step peak in the $l \to \infty$ limit.

Solving the continuity equations, $\psi_{\mathrm{I}}(0) = \psi_{\mathrm{II}}(0)$ and $\psi_{\mathrm{II}}(l) = \psi_{\mathrm{III}}(l)$, in matrix form yields

$$\begin{aligned} &\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{bmatrix} 1 \\ R_{>} \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ \alpha & -\alpha \end{pmatrix} \begin{bmatrix} A_{>} \\ B_{>} \end{bmatrix}, \\ &\begin{pmatrix} rr\mathrm{e}^{-ql} & \mathrm{e}^{ql} \\ \alpha\mathrm{e}^{-ql} & -\alpha\mathrm{e}^{ql} \end{pmatrix} \begin{bmatrix} A_{>} \\ B_{>} \end{bmatrix} = \begin{bmatrix} T_{>} \\ T_{>} \end{bmatrix} \mathrm{e}^{\mathrm{i}pl}\,, \end{aligned} \tag{6}$$

where

$$\alpha = \mathrm{i}\frac{q}{p}\frac{E+m}{E - V_0 + m}\,.$$

Since Dirac is a first order equation, only the continuity of ψ is imposed. Solving the above equations gives

$$\begin{aligned} \begin{bmatrix} 1 \\ R_{>} \end{bmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ \alpha & -\alpha \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-ql} & \mathrm{e}^{ql} \\ \alpha\mathrm{e}^{-ql} & -\alpha\mathrm{e}^{ql} \end{pmatrix}^{-1} \\ &\quad \times \begin{bmatrix} T_{>} \\ T_{>} \end{bmatrix} \mathrm{e}^{\mathrm{i}pl} \\ &= \frac{1}{2} \begin{bmatrix} \cosh(ql) + \alpha\sinh(ql) & \cosh(ql) + \frac{\sinh(ql)}{\alpha} \\ \cosh(ql) - \alpha\sinh(ql) & -\cosh(ql) + \frac{\sinh(ql)}{\alpha} \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} T_{>}\mathrm{e}^{\mathrm{i}pl}\,. \end{aligned} \tag{7}$$

Thus,

$$\begin{aligned} R_{>} &= \frac{1-\alpha^2}{2\alpha}\sinh(ql)T_{>}\exp[\mathrm{i}pl]\,, \\ T_{>} &= \exp[-\mathrm{i}pl] \Big/ \left[\cosh(ql) + \frac{1+\alpha^2}{2\alpha}\sinh(ql)\right]. \end{aligned} \tag{8}$$

The non-relativistic limit, $E-m=E_{\rm NR}\ll m$ and $V_0 \ll m$, reproduces the Schrödinger results for the reflection and transmission coefficients (see the appendix for a detailed derivation).

2.2 B zone: $V_0-m<E<V_0$ ($V_0>2m$) or $m<E<V_0$ ($m<V_0<2m$)

For this zone, in the potential region, we shall use the spinor $u^{(3)}$. Thus, we have

$$\begin{aligned}
&\text{Region I}: z<0\,,\\
&u^{(1)}(p,E)\exp[\mathrm{i}pz]+R_<u^{(1)}(-p,E)\exp[-\mathrm{i}pz]\,,\\
&\text{Region II}: 0<z<l\,,\\
&A_<u^{(3)}(\mathrm{i}q,|E-V_0|)\exp[-qz]+B_<u^{(3)}(-\mathrm{i}q,|E-V_0|)\\
&\quad\times\exp[qz]\,,\\
&\text{Region III}: l<z\,,\\
&T_<u^{(1)}(p,E)\exp[\mathrm{i}pz]\,.
\end{aligned}\tag{9}$$

The continuity equations in matrix form yield

$$\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{bmatrix}1\\R_<\end{bmatrix}=\frac{E+m}{p}\begin{pmatrix}-\beta&\beta\\1&1\end{pmatrix}\begin{bmatrix}A_<\\B_<\end{bmatrix}\tag{10}$$

and

$$\frac{E+m}{p}\begin{pmatrix}-\beta\mathrm{e}^{-ql}&\beta\mathrm{e}^{ql}\\\mathrm{e}^{-ql}&\mathrm{e}^{ql}\end{pmatrix}\begin{bmatrix}A_<\\B_<\end{bmatrix}=\begin{bmatrix}T_<\\T_<\end{bmatrix}\mathrm{e}^{\mathrm{i}pl}\,,\tag{11}$$

where

$$\beta=\mathrm{i}\frac{qp}{(E+m)(|E-V_0|+m)}\,.$$

Solving the above matrix equations, we find

$$\begin{aligned}
\begin{bmatrix}1\\R_<\end{bmatrix}&=\begin{pmatrix}1&1\\1&-1\end{pmatrix}^{-1}\begin{pmatrix}-\beta&\beta\\1&1\end{pmatrix}\begin{pmatrix}-\beta\mathrm{e}^{-ql}&\beta\mathrm{e}^{ql}\\\mathrm{e}^{-ql}&\mathrm{e}^{ql}\end{pmatrix}^{-1}\begin{bmatrix}T_<\\T_<\end{bmatrix}\mathrm{e}^{\mathrm{i}pl}\\
&=\frac{1}{2}\begin{bmatrix}\cosh(ql)-\frac{\sinh(ql)}{\beta}&\cosh(ql)-\beta\sinh(ql)\\\cosh(ql)+\frac{\sinh(ql)}{\beta}&-\cosh(ql)-\beta\sinh(ql)\end{bmatrix}\\
&\quad\times\begin{bmatrix}1\\1\end{bmatrix}T_>\mathrm{e}^{\mathrm{i}pl}\,.
\end{aligned}\tag{12}$$

Thus,

$$\begin{aligned}
R_<&=\frac{1-\beta^2}{2\beta}\sinh(ql)T_<\exp[\mathrm{i}pl]\,,\\
T_<&=\exp[-\mathrm{i}pl]\Big/\left[\cosh(ql)-\frac{1+\beta^2}{2\beta}\sinh(ql)\right]\,.
\end{aligned}\tag{13}$$

Although it may not be obvious, simple algebraic calculations show that $R_<$ and $T_<$ are functionally identical to $R_>$ and $T_>$ respectively (although, of course, valid in disjoint energy zones). Hence, in the following, we will drop the suffixes and use

$$\begin{aligned}
T&=\exp\left\{-\mathrm{i}pl+\mathrm{i}\arctan\left[\frac{E^2-m^2-EV_0}{qp}\tanh(ql)\right]\right\}\Big/\\
&\qquad\sqrt{\cosh^2(ql)+\left[\frac{E^2-m^2-EV_0}{qp}\sinh(ql)\right]^2}\,,\\
R&=-\mathrm{i}\frac{mV_0}{qp}\sinh(ql)T\exp[\mathrm{i}pl]\,.
\end{aligned}\tag{14}$$

3 Time analysis

In this section, we shall calculate the analytic expression of the transition times by using the SPM and, by numerical calculation, we present the appearance of superluminal velocities.

3.1 SPM transition times

For this analysis the essential ingredient is the phase of the transmitted amplitude T. The gaussian envelope function $g(p)$ will be assumed to be real, and the wave packet function $\Psi(x,t)$, defined in the standard way, is given by

$$\Psi(x,t)=\int\mathrm{d}pg(p)\psi(z)\exp[-\mathrm{i}Et]\,.\tag{15}$$

A common choice for $g(p)$ (unnormalized) peaked at p_0 is

$$g(p)=\exp\left[-a^2(p-p_0)^2/4\right]\,,$$

so that for the incoming wave $\psi_{\rm I,inc}(z)=u^{(1)}(p,E)\exp[\mathrm{i}pz]$, the wave packet width is just a (large barriers are thus defined by $l/a\gg1$). Due to the real nature of $g(p)$, the phase in $\Psi_{\rm I,inc}(x,t)$ is simply

$$\phi_{\rm I,inc}=pz-Et\,.\tag{16}$$

The SPM then sets the maximum of the incident wave packet at time t at

$$z=\left[\frac{\mathrm{d}E}{\mathrm{d}p}\right]_0\,,\quad t=\frac{p_0}{E_0}t\,,$$

where E_0 is the energy corresponding to the peak momentum value of p_0. Therefore, the incoming wave packet maximum (ignoring interference effects with the reflected wave) reaches $z=0$ at time $t=0$.

The SPM calculation of the transmission time uses the phase factor of T obtained in the previous section,

$$\begin{aligned}
\phi_{\rm III,tra}&=\arctan\left[\frac{E^2-m^2-EV_0}{qp}\tanh(ql)\right]\\
&\quad+p(z-l)-Et\,.
\end{aligned}\tag{17}$$

Taking the derivative of $\phi_{\text{III,tra}}$ with respect to E and setting $z=l$, we find the following functional:

$$t_{\text{tra}}(E,l) = \left\{ \frac{2E-V_0}{qp} \left[1 + \left(\frac{E^2-m^2-EV_0}{qp} \right)^2 \right] \tanh(ql) + \frac{(E^2-m^2-EV_0)(V_0-E)l}{q^2 p \cosh^2(ql)} \right\} \Bigg/ \left\{ 1 + \left[\frac{E^2-m^2-EV_0}{qp} \tanh(ql) \right]^2 \right\} . \quad (18)$$

The exit time of a single transmitted wave packet is then given by $t_{\text{tra}}(\tilde{E}_0, l)$, where $\tilde{E}_0 = \sqrt{\tilde{p}_0^2 + m^2}$, and $\tilde{p}_0$ is the peak momentum of the transmitted wave packet, i.e. (neglecting the spinor momentum dependencies) the maximum of $g(p)|T|$ (see Figs. 1 and 3). If the barrier is short, then $\tilde{p}_0 \approx p_0$ because the gaussian dominates over the transmission amplitude. On the other hand, if we take $l \to \infty$, the functional $t_{\text{tra}}(E,l)$ greatly simplifies, giving

$$\tau_{\text{tra}}(E) := \lim_{l\to\infty} t_{\text{tra}}(E,l) = \frac{2E-V_0}{qp} , \quad (19)$$

which is independent of l. At first glance, this would mean unlimited tunneling velocities. Actually, this is often argued without taking into account the difference between p_0 and $\tilde{p}_0$. The exit time is given by $\tau(\tilde{E}_0)$ and not by $\tau(E_0)$. If, for example, we set $\tilde{E}_0$ to its maximum allowed value (compatible with tunneling) $\tilde{E}_0 = V_0 + m$, we find

$$\tau_{\text{tra}}[V_0+m] \to \infty ,$$

so that the tunneling velocities are *not* in general unlimited. However, since we shall find in the next section superluminal velocities for finite l, we shall not dwell upon

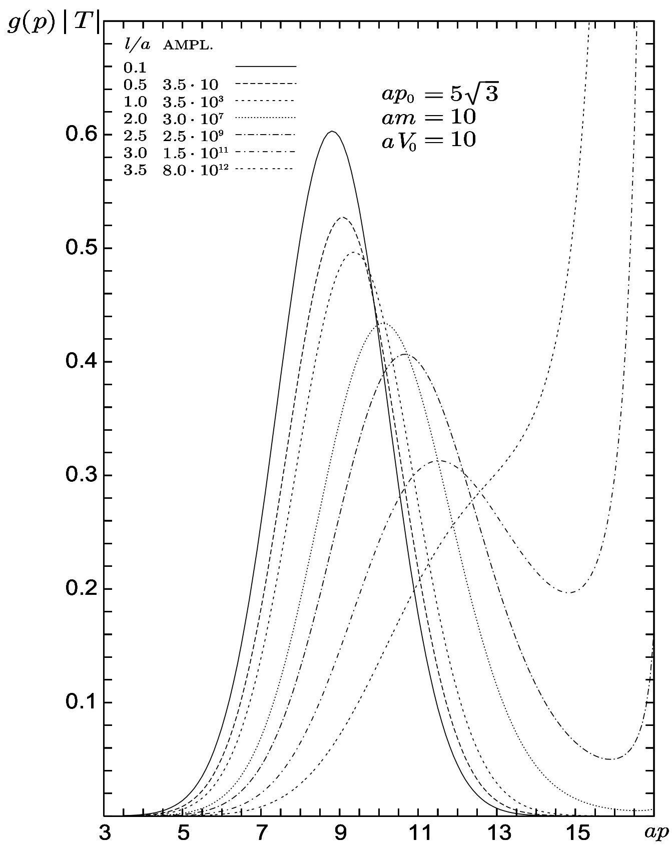

Fig. 1. The transmitted momentum distribution is plotted as a function of ap for different values of l/a, where a is the width of the incident wave packet at $t=0$. The potential is equal to the mass of the particle, $aV_0 = am = 10$, and the peak of the incident momentum distribution is chosen to coincide with the center of the allowed zone (compatible with tunnelling) for the momentum, $ap_0 = a\sqrt{V_0(V_0+2m)}/2 = 5\sqrt{3}$. For moderate values of l/a the transmitted momentum distribution is almost gaussian. The amplifications show the attenuation (due to the evanescent waves) of the transmission probability for increasing values of l/a

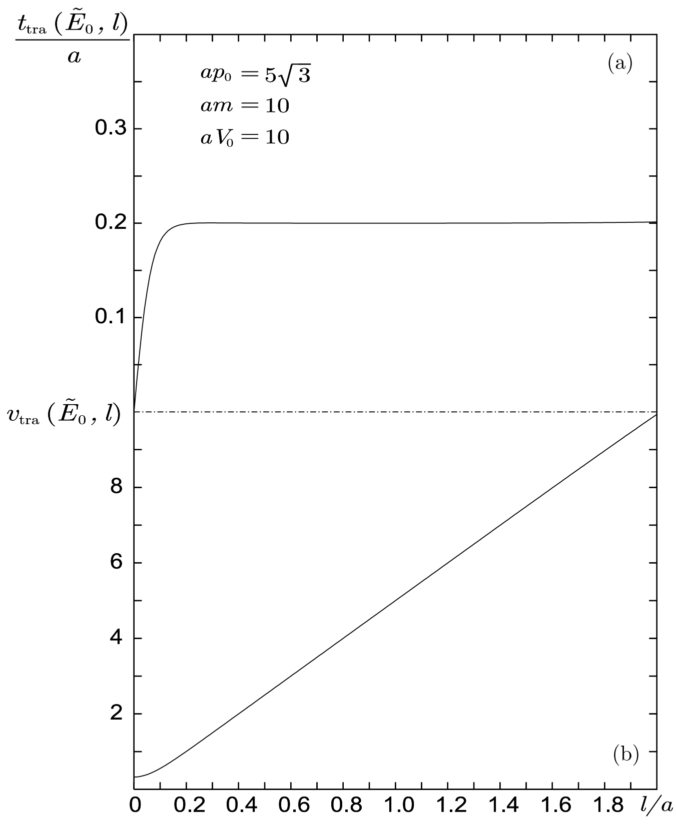

Fig. 2. This figure contains two curves. The plot in **a** represents the variation of the (adimensional) transmission time, $t_{\text{tra}}(\tilde{E}_0, l)/a$ as a function of l/a. The plot in **b** represents the ration between the barrier width, l, and the transmission time $t_{\text{tra}}(\tilde{E}_0, l)$, i.e. the effective velocity of the tunnelling process. The potential is equal to the mass of the particle, $aV_0 = am = 10$, and the peak of the incident momentum distribution is $ap_0 = 5\sqrt{3}$. The maximum value of l/a has been chosen to be 2.0 in order to have an almost gaussian transmitted wave packet. This guarantees the validity of the SPM. The surprising feature of our numerical analysis is that the tunnelling velocity is already greater than one for a moderate barrier width

the asymptotic tunneling velocity. We only note that if instead of a gaussian wave packet (which technically overshoots the tunneling zone) we use a truncated gaussian, we can avoid infinite SPM tunneling times by truncating below V_0+m. We warn, however, that a truncation in the momentum spectrum of a wave packet automatically introduces infinite wave packets in configuration space, and we have to take care in using the SPM [2, 5].

3.2 Superluminal velocities

In the previous subsection, we derived an expression for the transmission time,

$$t_{\rm tra}(\tilde{E}_0, l)\,.$$

This expression requires knowledge of $\tilde{E}_0$, the peak value of the transmitted momentum distribution. This implicitly assumes a single maximum. So, the SPM is certainly valid for moderate values of l, where the transmitted spectrum is almost gaussian as it is shown in Figs. 1 and 3 ($l \lesssim 2a$). In such a context, we have numerically calculated $\tilde{E}_0(l)$ and obtained $t_{\rm tra}(\tilde{E}_0, l)$ by (18). In Figs. 2a and 4a, we have plotted $t_{\rm tra}(\tilde{E}_0, l)/a$ against l/a. We observe that, for $l \gg a$, we have to use instead of the maximum momentum value the average value of the spectrum, i.e. $\tilde{E}_0 \equiv \langle E\rangle$. This is because for a momentum curve that ends on a maximum, thus being very asymmetric, numerical model calculations show that a more accurate result for the SPM times is obtained with the use of $\langle E\rangle$. The surprising feature of the curves given in Figs. 2a and 4a is obtained by taking the ratio of the coordinates,

$$v_{\rm tra}(\tilde{E}_0, l) = \frac{l}{t_{\rm tra}(\tilde{E}_0, l)}\,,$$

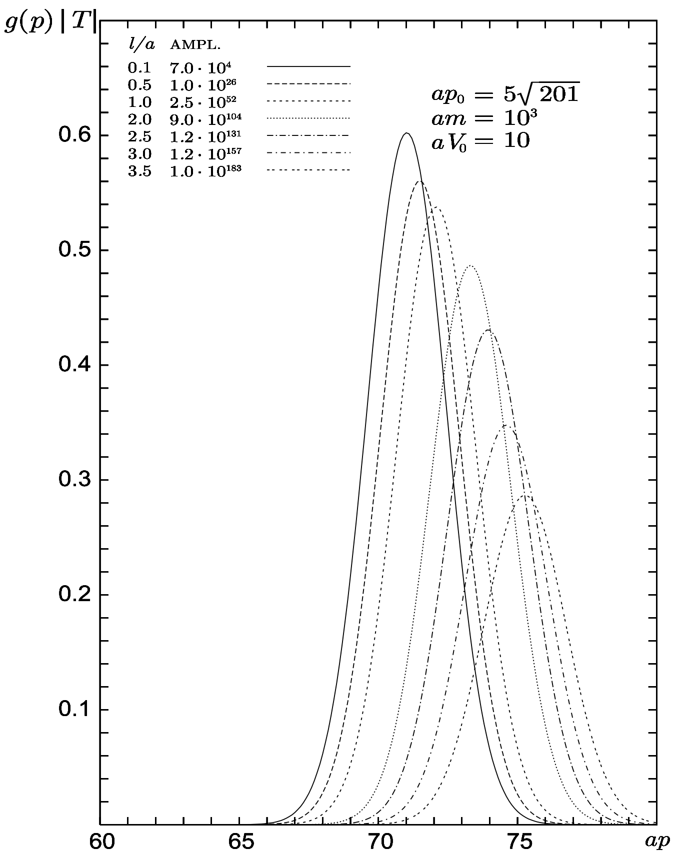

Fig. 3. The transmitted momentum distribution is plotted as a function of ap for different values of l/a. The potential $aV_0 = 10$ is now much smaller than the mass of the particle, $am = 10^3$, and the peak of the incident momentum distribution is yet chosen to coincide with the center of the allowed zone for the momentum, $ap_0 = am\sqrt{V_0(V_0+2m)}/2 = 5\sqrt{201}$. This case represents the non-relativistic limit, $V_0 \ll m$ and $E - m = E_{\rm NR} \ll m$. The amplifications show a considerable attenuation of the transmission probability with respect to the relativistic case

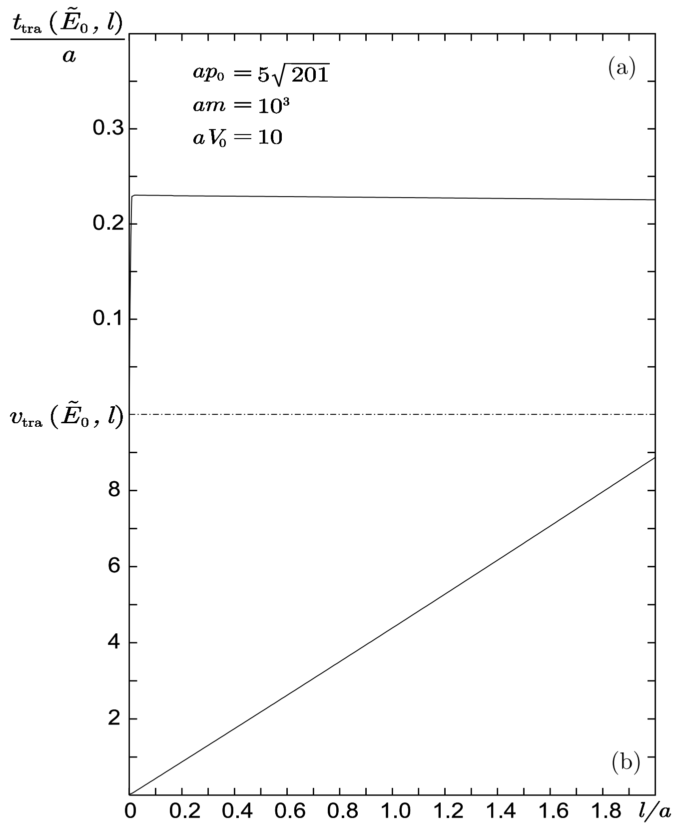

Fig. 4. The plot in **a** represents the variation of the (adimensional) transmission time, $t_{\rm tra}(\tilde{E}_0, l)/a$, as a function of l/a. The plot in **b** represents the ration between the barrier width, l, and the transmission time $t_{\rm tra}(\tilde{E}_0, l)$, i.e. the effective velocity of the tunneling process. The potential $aV_0 = 10$ is much smaller than the mass of the particle, $am = 10^3$, and the peak of the incident momentum distribution is $ap_0 = 5\sqrt{201}$. The maximum value of l/a has been chosen to be 2.0 in order to have an almost gaussian transmitted wave packet. This guarantees the validity of the SPM. The surprising feature of our numerical analysis, i.e. the tunneling velocity greater than one for a moderate barrier width, is confirmed in the non-relativistic limit

the effective velocity of the tunneling processes. This is plotted in Figs. 2b and 4b as a function of l/a. As it can be seen from Figs. 2a and 4a, there is a plateau region, where the transit time is independent of the value of l/a. In this region the effective velocity grows linearly (see Figs. 2b and 4b). However, the surprising feature is the numerical value of the velocity in this region: it is already greater than one both for the relativistic and for the non-relativistic case. We have no need to go to the infinite barrier width limit (Hartman effect) to find superluminal velocities.

4 Conclusions

We have studied in this paper the tunneling phenomena predicted by the Dirac equation. One of the principal questions asked at the start was if a Hartman effect exists also for the Dirac equation. The answer is positive, since the spinors play no significant role in the calculation of the transmission times. There is a difficulty with the fact that the different momentum dependence of the spinors lead to different transmission times for the components. However, this does not modify the result of each of them exhibiting a Hartman-like effect. It is the SPM that obliges us to work with wave function *amplitudes* rather than with the transmission probability function, in which spinor components have been summed over. However, if all spinor components yield superluminal velocities, then superluminal velocities must be expected. The Hartman limit ($l \gg a$) has an added complication, because the transmission function dominates the incoming convolution function (gaussian in all our calculations), and it is not yet clear how, or even if, the SPM works in the absence of a clean maximum in the momentum distribution. We have avoided entering into this equation, because there is no need to go to $l \gg a$ in order to exceed the velocity of light.

A second question involved in our study was the identification of the nature (charge) of the particles temporarily (for wave packets) in the classically forbidden barrier region. This is a relevant question when one recalls that in the Klein energy zone ($E < V_0 - m$) antiparticles are created and/or annihilated in the barrier region. For an antiparticle the barrier becomes a well, and the particles of energy E *mathematically* below the potential (V_0) are re-interpreted as antiparticles of energy $-E$ *physically* above the potential ($-V_0$) [6,9]. It is tempting to consider all "particles" with $E < V_0$ (below potential) to be in fact physical antiparticles, even if associated to evanescent terms (B tunneling zone). This is the reason why we divided the tunneling energy zone into two: the A ($V_0 < E < V_0 + m$) and the B ($V_0 - m < E < V_0$) zones. We now give an argument based upon our studies of the Klein zone that says that this hypothesis is *not* true.

Let us consider in the following the simple step potential. The Dirac equation conserves probabilities. How is this consistent with pair creation in which more particles are reflected, $R > 1$ (Klein paradox), than are incident? Physically, the total charge is conserved, but the probability certainly is not. The answer is suggested by the well known fact that the particles below potential in the Klein zone have the "wrong" group velocity. This fact incidentally concords with the Feynman–Stückelberg conclusion that such particles with energy below potential must actually travel backwards in time. Returning to our conundrum, we observe that in any formal numerical calculation that ignores the antiparticle re-interpretation, the Klein paradox appears mathematically as the particular solution to a problem in which at $t = -\infty$ we have *two opposite moving wave packets*: the incident wave packet at $z = -\infty$ and the wave packet below potential (of appropriate size) at $z = +\infty$. When these two meet at time $t \sim 0$ at the step discontinuity $z = 0$, the continuity equations tell us that they unite and form a single wave packet, the reflected wave packet [20]. In this way probability is indeed seen to be conserved. It is the re-interpretation in physical particle/antiparticle terms that alters our viewpoint. However, the Dirac equation (with a real potential) can be viewed in this mathematical picture with *only particles*, albeit with energies both above and below potential. Indeed this is the way everyone treats the stationary plane wave problem, including ourselves [6].

Let us now apply the same mathematical viewpoint to the step in the energy zones A and B. In this case, there cannot be any effective particle flow from $z = \infty$, since the stationary solution with the barrier is, in both energy zones, a pure exponentially decreasing space function. At time $t = -\infty$ only the incoming wave packet exists in this case. Eventually, for $t = +\infty$ only the reflected wave packet exists. For times within the transmission period during which complete reflection occurs, we cannot have a reflection coefficient $R > 1$, even if only for an instant, without violating probability conservation. This conclusion is independent of the choice of the A or B zones. It is based upon the impossibility of having a modification of the initial conditions so as to reproduce a transitory Klein-like paradox. Consequently, even in tunneling, the probability density under the potential must represent the same particles as those of the incoming wave.

It would be desirable to conclude the debate on superluminal velocities in tunneling phenomena, but, at the moment, the results are far from being conclusive. For a potential of the order of the mass, the Dirac equation cannot be viewed as a one-particle equation, and particle creation is expected to play an important role. There is still much to be studied in potential problems within field theory, and the final answer can only be reached by analyzing tunneling phenomena in a second quantized theory [21,22]. However, this topic exceeds the scope of this paper, and it will be appropriately discussed in a forthcoming article. In such a spirit, this paper has to be seen as an initial work to stimulate further investigations.

Acknowledgements. One of the authors (S.D.L.) wishes to thank the Department of Physics of Lecce University, where the paper was prepared, for invitation and hospitality. He also thanks the FAEP (Brazil) and the INFN (Italy) for financial support. The authors gratefully acknowledge the helpful suggestions and comments of an anonymous referee which allowed them to improve the presentation of this paper. In particular, they express their gratitude to the referee for drawing the attention to the analysis of potentials much smaller than the mass of the particle. This stimulated, in the revised version, a careful and detailed analysis of the non-relativistic limit.

Appendix

Let us discuss in detail the non-relativistic (NR) limit, $E-m=E_{\rm NR}\ll m$ and $V_0\ll m$. We recall that for $V_0<m$ only the evanescent zone (A), $0<E_{\rm NR}<V_0$, exists. For the convenience of the reader, we rewrite the Dirac reflection and transmission coefficients given in the text,

$$\begin{aligned} T &= \exp[-\mathrm{i}pl] \Big/ \left[\cosh(ql)+\frac{1+\alpha^2}{2\alpha}\sinh(ql)\right], \\ R &= \frac{1-\alpha^2}{2\alpha}\sinh(ql)\exp[\mathrm{i}pl], \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} p &= \sqrt{E^2-m^2}, \\ q &= \sqrt{m^2-(E-V_0)^2}, \\ \alpha &= \mathrm{i}\frac{q}{p}\frac{E+m}{E-V_0+m}. \end{aligned} \tag{A.2}$$

Taking the NR limit, we obtain

$$\begin{aligned} p &\to p_{\rm NR}=\sqrt{2mE_{\rm NR}}, \\ q &\to q_{\rm NR}=\sqrt{2m(V_0-E_{\rm NR})}, \\ \alpha &\to \mathrm{i}q_{\rm NR}/p_{\rm NR}. \end{aligned} \tag{A.3}$$

Consequently,

$$\begin{aligned} T_{\rm NR} &= \exp[-\mathrm{i}p_{\rm NR}l] \Big/ \\ &\quad\times\left[\cosh(q_{\rm NR}l)-\mathrm{i}\frac{2E_{\rm NR}-V_0}{2\sqrt{E_{\rm NR}(V_0-E_{\rm NR})}}\sinh(q_{\rm NR}l)\right], \\ R_{\rm NR} &= -\mathrm{i}\frac{V_0}{2\sqrt{E_{\rm NR}(V_0-E_{\rm NR})}}\sinh(q_{\rm NR}l)T_{\rm NR} \\ &\quad\times\exp[\mathrm{i}p_{\rm NR}l]. \end{aligned} \tag{A.4}$$

The square modulus of these coefficients,

$$\begin{aligned} |T_{\rm NR}|^2 &= \\ &\frac{4E_{\rm NR}(V_0-E_{\rm NR})}{4E_{\rm NR}(V_0-E_{\rm NR})+V_0^2\sinh^2\left[\sqrt{2m(V_0-E_{\rm NR})}l\right]}, \\ |R_{\rm NR}|^2 &= \\ &\frac{V_0^2}{4E_{\rm NR}(V_0-E_{\rm NR})}\sinh^2\left[\sqrt{2m(V_0-E_{\rm NR})}l\right]|T_{\rm NR}|^2, \end{aligned} \tag{A.5}$$

is often encountered in standard quantum mechanics textbooks (see for example [23]).

References

1. S. De Leo, P. Rotelli, Phys. Rev. A **70**, 022 101 (2004)
2. A.E. Bernardini, S. De Leo, P. Rotelli, Mod. Phys. Lett. A **19**, 2717 (2004)
3. S. De Leo, P. Rotelli, Phys. Lett. A **342**, 294 (2005)
4. S. De Leo, P. Rotelli, Phys. Rev. D **69**, 034 006 (2004)
5. S. De Leo, P. Rotelli, Eur. Phys. J. C **46**, 551 (2006)
6. S. De Leo, P. Rotelli, Phys. Rev. A **73**, 042 107 (2006)
7. O. Klein, Z. Phys. **53**, 157 (1929)
8. A. Hansen, F. Ranvndal, Phys. Scripta **23**, 1036 (1981)
9. B.R. Holstein, Am. J. Phys. **66**, 507 (1998)
10. E.P. Wigner, Phys. Rev. **98**, 145 (1955)
11. T.E. Hartman, J. Appl. Phys. **33**, 3427 (1962)
12. P. Krekora, Q. Su, R. Grobe, Phys. Rev. A **63**, 032 107 (2001)
13. P. Krekora, Q. Su, R. Grobe, Phys. Rev. A **64**, 022 105 (2001)
14. V. Petrillo, D. Janner, Phys. Rev. A **67**, 012 110 (2003)
15. A.I. Baz, Sov. J. Nucl. Phys. **4**, 182 (1967)
16. M. Büttiker, R. Laundayer, Phys. Rev. Lett. **49**, 1739 (1982)
17. E. Pollak, W.H. Miller, Phys. Rev. Lett. **53**, 115 (1984)
18. G. Nimtz, W. Heitmann, Prog. Quantum Electron. **21**, 81 (1997)
19. V.S. Olkhovsky, E. Recami, J. Jakiel, Phys. Rep. **398**, 133 (2004)
20. H. Nitta, T. Kudo, H. Minowa, Am. J. Phys. **67**, 966 (1999)
21. M.E. Perskin, D.V. Schroeder, An Introduction to Quantum Field Theory (Perseus Books, Cambridge, Massachusetts, 1995)
22. F. Gross, Relativistic Quantum Mechanics and Field Theory (John Wiley and Sons, New York, 1999)
23. C. Choen-Tannoudji, B. Diu, F. Lalöe, Quantum Mechanics (Wiley, New York, 1977), p. 73